VAM Applied to Dimensional Reduction of Nonlinear Multifunctional Film-fabric Laminates

Alap Kshirsagar*, Dineshkumar Harursampath† and Ramesh Gupta Burela**

*Department of Mechanical Engineering, Indian Institute of Technology, Bombay, INDIA
†Nonlinear Multifunctional Composites Analysis and Design Lab, Department of Aerospace Engineering, Indian Institute of Science, Bangalore, INDIA
**Mechanical Engineering Department, Shiv Nadar University, Delhi NCR, INDIA

Abstract. This work aims at asymptotically accurate dimensional reduction of non-linear multi-functional film-fabric laminates having specific application in design of envelopes for High Altitude Airships (HAA). The film-fabric laminate for airship envelope consists of a woven fabric core coated with thin films on each face. These films provide UV protection and Helium leakage prevention, while the core provides required structural strength. This problem is both geometrically and materially non-linear. To incorporate the geometric non-linearity, generalized warping functions are used and finite deformations are allowed. The material non-linearity is handled by using hyper-elastic material models for each layer. The development begins with three-dimensional (3-D) nonlinear elasticity and mathematically splits the analysis into a one-dimensional through-the-thickness analysis and a two-dimensional (2-D) plate analysis. The through-the-thickness analysis provides the 2-D constitutive law which is then given as an input to the 2-D reference surface analysis. The dimensional reduction is carried out using Variational Asymptotic Method (VAM) for moderate strains and very small thickness-to-wavelength ratio. It features the identification and utilization of additional small parameters such as ratio of thicknesses and stiffness coefficients of core and films. Closed form analytical expressions for warping functions and 2-D constitutive law of the film-fabric laminate are obtained.

Keywords: VAM, film-fabric laminates, dimensional reduction, constitutive law
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INTRODUCTION

Stratospheric airships have generated great interest in multi role capabilities that include, cost effective alternative to earth satellites for telecommunication and science observation and cost effective payload capability with low fuel consumption[1]. To meet these requirements with high rate of performance a wider scope is available on modeling techniques in the form of multi functional composite materials. These structures are meant for low helium permeability, protection from UV radiation apart from the primary structural function of providing strength. Stratospheric airship envelopes consist of a woven fabric core covered with thin protective films on each face referred as film-fabric laminates. Typically Poly-Vinyl-Fluoride films (like teflon, tedlar etc.) are used for UV protection and Helium leakage prevention. The load carrier or core is made from laminated fibre yarns (vectran or zylon)[3]. Current work focuses on the analytical development of nonlinear constitutive law and warping functions in an asymptotically accurate form using a mathematical methodology Variational Asymptotic Method (VAM) introduced by Berdichevsky [5].

ANALYSIS

The development begins with three-dimensional nonlinear elasticity and mathematically splits the analysis into a one-dimensional through-the-thickness analysis and a two-dimensional plate analysis. The through-the-thickness analysis provides the 2-D constitutive law which is then given as an input to the 2-D reference surface analysis. The dimensional reduction is carried out using VAM for moderate strains and very small thickness-to-wavelength ratio. During this transformation, the additional small parameters pertaining to the ratio of the thickness of the surface films to the thickness of the woven fabric core and the stiffness of surface films to the stiffness of the woven fabric core are utilized to ensure that all contributions of the same order to the total energy from different layers of the laminate are accounted at each stage of applying VAM. The formulation does not involve any *ad hoc* kinematic assumptions or correction factors.

The typical film-fabric laminate structure for airship envelopes is shown in Figure(1). The film-fabric laminate is considered to be made up of three laminae with central woven fabric core and one thin Poly-Vinyl-Fluoride film each at the top and bottom. In the present formulation all possible deformations (large displacements and rotations) are accounted by using Green Strain tensor. To incorporate the material non-linearity, the strain energy of the core is obtained using orthotropic hyperelastic Saint Venant Kirchoff (SVK) 3-D strain energy function. Similarly the strain energies of both the top and bottom films are obtained using isotropic hyperelastic SVK 3-D strain energy function.

The warping field in each lamina of the film-fabric laminate is subjected to following global constraints as suggested by [6]:

$$\langle w_i(x_1,x_2,x_3)\rangle = 0 \quad ; \quad \langle x_3 w_\alpha(x_1,x_2,x_3)\rangle = 0 \tag{1}$$

where angular brackets denote the integration over the thickness of the film-fabric laminate structure at any given point on the mid-surface plane. Though the warping field of any normal to the mid-surface of the structure is allowed different functional forms within each layer of the film-fabric laminate, the continuity conditions at the interfaces of the layers have to be satisfied. The continuity conditions are mathematically described as:

$$w_i^t = w_i^c \Big|_{x_3=\frac{t_c}{2}}; \quad w_i^b = w_i^c \Big|_{x_3=\frac{-t_c}{2}}; \quad S_{i3}^t = S_{i3}^c \Big|_{x_3=\frac{t_c}{2}}; \quad S_{i3}^b = S_{i3}^c \Big|_{x_3=\frac{-t_c}{2}} \tag{2}$$

where S_{i3}^t, S_{i3}^c and S_{i3}^b are the transverse stress components of 2^{nd} Piola Kirchoff Stresses in the top, core and bottom layers of the film-fabric laminate respectively.

The solutions of warping functions are the stationary points of total potential energy functional, w.r.t. the 3-D warping variables, subjected to the global constraints and interlaminar continuity constraints. Variational Asysmptotic Method (VAM), which is a synergy of variational and asymptotic procedures, enables us to obtain asymptotically accurate solutions of the warping fuctions by making use of certain small parameters inherent to the problem definition

In order to apply VAM, we first need to assess the order of magnitudes of all the quantities which are contributing the energy. Order of magnitude of small parameters are $\frac{t}{l} \sim O(\delta^4),\ \frac{t_c}{t_f} \sim O(\delta),\ \Gamma_{ij} \sim O(\delta^2),\ \frac{\lambda}{a_{ij}} \sim O(\delta),\ \frac{\mu}{G_{ij}} \sim O(\delta)$ where t, t_c and t_f are the thicknesses of laminate, core and films respectively, l is the characteristic length (the wavelength of plate), λ and μ are the stiffness co-efficients of the top and bottom layers, a_{ij} and G_{ij} are the stiffness co-efficients of the core. The fact that the maximum 3-D strain component anywhere in the plate is of $O(\delta_\varepsilon)$ helps in estimation of the 2-D strains, warping functions and its derivatives are given below: $\varepsilon_{\alpha\beta} \sim O(\delta^2),\ \kappa_{\alpha\beta} \sim O(\frac{1}{l\delta^2}),\ \gamma_{\alpha 3} \sim O(\delta^2),\ w_i \sim O(l\delta^6),\ w_{i,\alpha} \sim O(\delta^6),\ w_{i,3} \sim O(\delta^2)$. Thus the total energy expression can be categorized into different orders of magnitude such that: $\Pi = \Pi_0 + \Pi_1 + \Pi_2 \ldots$, where $\Pi_0 \gg \Pi_1 \gg \Pi_2 \ldots$ Here Π_0 is the zeroth order energy, Π_1 is the first order energy and so on. The higher order energy terms are less critical for engineering applications.

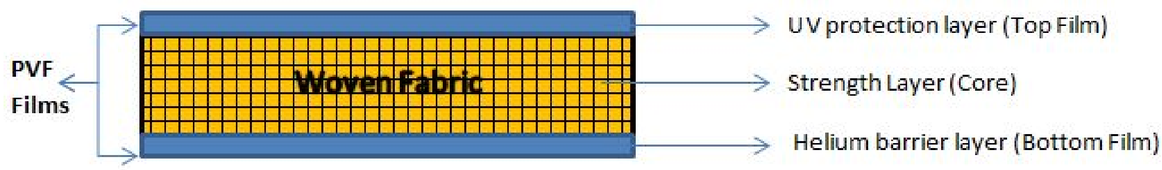

FIGURE 1. Film-fabric laminate structure

Zeroth order approximation

The zeroth-order approximation should contain all terms up to $O(\delta^2)$ in the 3-D strain expression. The global constraints on the warping field listed in Equation(1) are now introduced into the formulated zeroth-order approximation.

This is carried out by applying the usual procedure of the calculus of variations with the aid of Lagrange multipliers.
Zeroth Order Warping Functions:

- Warping field of top film, core and bottom film in x_1 direction:

$$w_1^t = \frac{\gamma_{13}(\mu-G_{13})t_c^3}{12G_{13}(t_c^2+5t_ct_f+5t_f^2)},\quad w_1^c = \frac{\gamma_{13}(\mu-G_{13})\left(12x_3+\frac{5(4x_3^3-3x_3t_c^2)(t_c^2+6t_ct_f+6t_f^2)}{t_c^2(t_c^2+5t_ct_f+5t_f^2)}\right)}{12G_{13}},\quad w_1^b = \frac{\gamma_{13}(-\mu+G_{13})t_c^3}{12G_{13}(t_c^2+5t_ct_f+5t_f^2)}$$

- Similarly, warping field of top film, core and bottom film in x_2 direction:

$$w_2^t = \frac{\gamma_{23}(\mu-G_{23})t_c^3}{12G_{23}(t_c^2+5t_ct_f+5t_f^2)}\; w_2^c = \frac{\gamma_{23}(\mu-G_{23})\left(12x_3+\frac{5(4x_3^3-3x_3t_c^2)(t_c^2+6t_ct_f+6t_f^2)}{t_c^2(t_c^2+5t_ct_f+5t_f^2)}\right)}{12G_{23}}\; w_2^b = \frac{\gamma_{23}(-\mu+G_{23})t_c^3}{12G_{23}(t_c^2+5t_ct_f+5t_f^2)}$$

- Warping field of top film, core and bottom film in x_3 direction:

$$w_3^t = \frac{t_c\left((\varepsilon_{11}+\varepsilon_{22})\lambda-\varepsilon_{11}a_{13}-\varepsilon_{22}a_{23}+\frac{((\kappa_{11}+\kappa_{22})\lambda-\kappa_{11}a_{13}-\kappa_{22}a_{23})t_c^2}{6(t_c+2t_f)}\right)}{2a_{33}}$$

$$w_3^c = \frac{12x_3t_c((2(\varepsilon_{11}+\varepsilon_{22})+(\kappa_{11}+\kappa_{22})x_3)\lambda-(2\varepsilon_{11}+\kappa_{11}x_3)a_{13}-(2\varepsilon_{22}+\kappa_{22}x_3)a_{23})}{24a_{33}\left(t_c+2t_f\right)}+\frac{t_c^3(-(\kappa_{11}+\kappa_{22})\lambda+\kappa_{11}a_{13}+\kappa_{22}a_{23})}{24a_{33}\left(t_c+2t_f\right)}$$
$$+\frac{24t_fx_3((2(\varepsilon_{11}+\varepsilon_{22})+(\kappa_{11}+\kappa_{22})x_3)\lambda-(2\varepsilon_{11}+\kappa_{11}x_3)a_{13}-(2\varepsilon_{22}+\kappa_{22}x_3)a_{23})}{24a_{33}\left(t_c+2t_f\right)}+\frac{6t_ft_c^2(-(\kappa_{11}+\kappa_{22})\lambda+\kappa_{11}a_{13}+\kappa_{22}a_{23})}{24a_{33}\left(t_c+2t_f\right)}$$

$$w_3^b = \frac{t_c\left(-(\varepsilon_{11}+\varepsilon_{22})\lambda+\varepsilon_{11}a_{13}+\varepsilon_{22}a_{23}+\frac{((\kappa_{11}+\kappa_{22})\lambda-\kappa_{11}a_{13}-\kappa_{22}a_{23})t_c^2}{6(t_c+2t_f)}\right)}{2a_{33}}$$

In these expressions, ε_{11} and ε_{22} are 2-D in-plane strains along x_1 and x_2 respectively, γ_{12} is the in-plane shear strain, κ_{11} and κ_{22} are the reference surface bending curvatures about coordinate curves x_1 and x_2 respectively, $\gamma_{\alpha 3}$ are the transverse shear strains. These solutions, which are explicit in the thickness coordinate, x_3, and implicit in the in-plane coordinates, x_α, through the 2-D generalized strains, are substituted back into the zeroth-order energy expression Π_0 , and integrated over the thickness to get the 2-D strain energy density. Partial derivatives of the 2-D energy w.r.t. the 2-D strains yield the 2-D generalized constitutive law:

$$\begin{pmatrix} F \\ M \\ T \end{pmatrix} = \begin{bmatrix} A & B & 0 \\ B & D & 0 \\ 0 & 0 & E \end{bmatrix} \begin{pmatrix} \varepsilon \\ \kappa \\ \gamma \end{pmatrix} \tag{3}$$

where the force (F), moment (M) and transverse shear stress (T) resultants in the above equation correspond to the following:

$$F = \left(\frac{\partial \Pi}{\partial \varepsilon}\right)^T, M = \left(\frac{\partial \Pi}{\partial \kappa}\right)^T, T = \left(\frac{\partial \Pi}{\partial \gamma}\right)^T \tag{4}$$

In Equation(3), A, B, D and E are the membrane stiffness sub-matrix, coupling stiffness sub-matrix, bending stiffness sub-matrix and transverse shear stiffness sub-matrix respectively, given by:

$$A_{11} = \frac{\left(\lambda^2+\lambda a_{31}-a_{13}\left(\lambda+a_{31}\right)+a_{11}a_{33}\right)t_c}{a_{33}},\quad A_{22} = \frac{\left(\lambda^2+\lambda a_{32}-a_{23}\left(\lambda+a_{32}\right)+a_{22}a_{33}\right)t_c}{a_{33}},$$
$$A_{12} = A_{21} = \frac{\left(2\lambda^2+\lambda a_{31}-a_{23}\left(\lambda+a_{31}\right)+\lambda a_{32}-a_{13}\left(\lambda+a_{32}\right)+a_{12}a_{33}+a_{21}a_{33}\right)t_c}{2a_{33}},$$
$$A_{33} = \frac{\left(G_{12}+G_{21}\right)t_c}{2},\quad D_{11} = \frac{\left(\lambda^2+\lambda a_{31}-a_{13}\left(\lambda+a_{31}\right)+a_{11}a_{33}\right)t_c^3}{12a_{33}},$$
$$D_{12} = D_{21} = \frac{\left(2\lambda^2+\lambda a_{31}-a_{23}\left(\lambda+a_{31}\right)+\lambda a_{32}-a_{13}\left(\lambda+a_{32}\right)+a_{12}a_{33}+a_{21}a_{33}\right)t_c^3}{24a_{33}}$$
$$D_{22} = \frac{\left(\lambda^2+\lambda a_{32}-a_{23}\left(\lambda+a_{32}\right)+a_{22}a_{33}\right)t_c^3}{12a_{33}},\quad D_{33} = \frac{\left(G_{12}+G_{21}\right)t_c^3}{24}$$

$$E_{11} = \frac{(G_{13}+G_{31})\,t_c\left(\left(\mu^2+5G_{13}^2\right)t_c^4+10\left(\mu^2-\mu G_{13}+6G_{13}^2\right)t_c^3t_f+10\left(4\mu^2-7\mu G_{13}+24G_{13}^2\right)t_c^2t_f^2\right)}{12G_{13}^2\left(t_c^2+5t_ct_f+5t_f^2\right)^2)} \tag{5}$$

$$+\frac{(G_{13}+G_{31})\,t_c\left(60\left(\mu^2-2\mu G_{13}+6G_{13}^2\right)t_ct_f^3+30\left(\mu^2-2\mu G_{13}+6G_{13}^2\right)t_f^4\right)}{12G_{13}^2\left(t_c^2+5t_ct_f+5t_f^2\right)^2}$$

$$E_{22} = \frac{(G_{23}+G_{32})\,t_c\left(\left(\mu^2+5G_{23}^2\right)t_c^4+10\left(\mu^2-\mu G_{23}+6G_{23}^2\right)t_c^3t_f+10\left(4\mu^2-7\mu G_{23}+24G_{23}^2\right)t_c^2t_f^2\right)}{12G_{23}^2\left(t_c^2+5t_ct_f+5t_f^2\right)^2} \tag{6}$$

$$+\frac{(G_{23}+G_{32})\,t_c\left(60\left(\mu^2-2\mu G_{23}+6G_{23}^2\right)t_ct_f^3+30\left(\mu^2-2\mu G_{23}+6G_{23}^2\right)t_f^4\right)}{12G_{23}^2\left(t_c^2+5t_ct_f+5t_f^2\right)^2}$$

First Order Approximation

The next higher order contributions, in the asymptotic sense, to the warping fields are calculated through a procedure identical to the previous step but applied to the refined total energy. This is carried out by incorporating the next higher order terms in the strain tensor so that the total potential energy of the structure is better approximated by the additional energy contribution due to the inclusion of the higher order terms of the strain tensor. The analytical expressions of first order warping functions and stiffness sub-matrices are not included in this extended abstract due to space constraints.

CONCLUSION

This paper presented the use of Variational Asymptotic Method to develop asymptotically accurate two dimensional (2-D) constitutive law for film-fabric laminates with specific applications in stratospheric airship envelopes. The formulation is valid for moderate strains and small ratio of thickness to maximum wavelength of mid-surface deformation. It incorporates the geometric and material non-linearities and does not involve any kinematic assumption or correction factors. The contribution of this work is the derivation of closed form analytical expressions for warping functions and 2-D constitutive law of the film- fabric laminate, along with a set of recovery relations to express approximately the 3-D displacement, strain and stress fields. The 2-D constitutive law can be used to carry out various types of analyses like static, dynamic, stability etc. Moreover the use of additional small parameters pertaining to the geometric and physical properties of woven fabric core and protective films in a film-fabric laminate provide a new strategy in the application of VAM to problems of dimensional reduction.

REFERENCES

1. Kang W., Suh Y., Woo K. and Lee I., 2006, “Mechanical property characterization of film-fabric laminate for stratospheric airship envelope”, *Composite Structures*, **75**, pp. 151-155.
2. Liao L. and Pasternak I., “A review of airship structural research and development”, 2009, *Progress in Aerospace Sciences*, **45**, pp. 83-96.
3. High Altitude Airships: Detailed Project Report http://www.casde.iitb.ac.in/haa/CritTech/Material.pdf
4. Reddy J.N. , ‘Theory and Analysis of elastic plates and shells’, 2007, CRC press, Taylor and Francis.
5. Berdichevsky V.L., "Variational asymptotic method of shell theory construction", 1979, *Journal of Applied Mathematics and Mechanics*, **43**(4), pp. 711-736.
6. Atilgan A.R., Hodges D.H., 1992, “On the strain energy of laminated composite plates”, *International Journal of Solids and Structures*, **29**(20), pp. 2527-2543.
7. Rao M.V.P., Harursampath D., Renji K., “Stress analysis of composite honeycomb sandwich panels using the variational asymptotic method”, 2009, ISAMPE National Conference on Composites (INCCOM-8), Thiruvanathapuram, India, pp. 196-204.
8. Rao M.V.P., Harursampath D., Renji K., 2012, “Prediction of inter-laminar stresses in composite honeycomb sandwich panels under mechanical loading using Variational Asymptotic Method”, *Composite Structures*, **94**, pp. 2523-2537.

9. Berdichevsky V.L., "An asymptotic theory of sandwich plates", 2010, *International Journal of Engineering Science*, **48**(3):383-404.
10. Yieng Z., Lei C. and Yu W., "Variational asymptotic modeling of the multilayer functionally graded cylindrical shells ", 2012, *Composite Structures*, **94**, 966-976.
11. Burela R., Harursampath D., "VAM applied to dimensional reduction of non-linear hyperelastic plates", 2012, *International Journal of Engineering Science*, **59**, 90-102.
12. Malvern L. E., "Introduction to the mechanics of a continuous medium", 1969, Prentice-Hall, Inc.